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ON THE LOOKBACK DISTORTION RISK MEASURE: THEORY AND APPLICATIONS

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Abstract

Within the set of financial losses with equal means and variances, a sound coherent distortion risk measure should preserve some higher degree stop-loss order, e.g., the degree three convex order. Risk measures that satisfy this property are called tail-free risk measures. Restricting the set of terminal values of martingale financial losses to biatomic, Weibull and Pareto losses, we show that a specific distortion measure is a tail-free coherent measure and satisfies a meaningful extra condition used to measure the risk of such financial losses. This main result is applied to derive an optimal economic capital formula for lookback financial losses. It is used to compare the riskiness of two investment strategies.

1. Introduction

The axiomatic approach to risk measures is an important topic of financial mathematics, which finds applications in actuarial science (premium calculation), finance (portfolio selection), and risk management (capital requirements). Besides the *coherent risk measures* by Artzner et al. [1, 2], one is interested in the *distortion risk measures* by Denneberg

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[4, 5], Wang [23, 24], and Wang et al. [27]. Under certain circumstances, distortion measures are coherent risk measures (e.g., Wang et al. [27], Theorem 3). For this reason, they can be used to determine the capital requirements of a risky business, as suggested by several authors including Wirch and Hardy [28], Wang [26], and Goovaerts et al. [8].

Despite of being coherent, there exist distortion measures, which do not always provide incentive for risk management, because they lack to give a capital relief in some simple two scenarios situations of reduced risk (see Example 2.1 for the conditional value-at-risk measure). With equal means and variances of risks, one is interested in distortion measures, which preserve the higher degree stop-loss orders. Indeed, it is known that a coherent distortion measure preserves the usual stochastic order and the stop-loss order. This is a desirable property because increased risk should be penalized with an increased measure. With equal means and variances, a stop-loss order relation between different random variables cannot exist. In this situation, increased risk can be modelled by the degree three stop-loss order, or equivalently, by equal mean and variance, the degree three convex order. Thus, one is interested in distortion measures, which preserve this higher degree convex order. Such measures are called free of tail risk or simply *tail-free distortion measures*. The construction of degree two coherent tail-free risk measures has been discussed in Hürlimann [15], where the Wang right-tail measure has been justified on an axiomatic basis and applied to the derivation of an optimal economic capital formula. The present contribution displays the lookback distortion measure as a further possible degree two tail-free coherent distortion measure for use in the context of lookback financial losses. A more detailed account of the paper follows.

In Section 2, we recall the notions of coherent risk measure and distortion measure and identify the distortion measures, which induce coherent risk measures. Then, we define the notion of a higher degree tail-free risk measure by requiring the preservation property under a higher degree convex order. Example 2.1 illustrates the relevance of this notion in risk management. In Section 3, the evolution of a financial loss over a time period of length one is modelled by a *martingale* $\{X(t)\}_{0\leq t\leq 1}$, where the terminal random value $X = X(1)$ follows a given distribution. Then the possible maxima of martingales $M = \sup_{0\leq t\leq 1}\{X(t)\}$ with fixed $X = X(1)$ coincide with the set of random variables stochastically bounded below by X and above by the Hardy-Littlewood transform X^H of X. This important and useful tool, well-known in the mathematical literature, is summarized in our Theorem 3.1, called here *BDGKR Theorem* (read Blackwell-Dubins-Gilat-Kertz-Rösler Theorem). Using the axiomatic characterization of the coherent distortion measures by Wang et al. [27] and adding some extra condition, we show in Section 4 that the lookback distortion measure, defined by $g(t) = t^{\rho}\cdot[1-\rho\cdot\ln(t)]$, $\rho\in(0, 1]$, is a meaningful coherent distortion measure for use in the context of lookback financial losses. This measure is degree two tail-free for the subset of biatomic losses, the Weibull and Pareto losses by equal finite means and variances if, and only if, one has $\rho = \frac{1}{2}$, as shown in the Propositions 5.2 and 5.4. Together, this yields in Theorem 5.1 a characterization of the lookback distortion measure defined by $g(t) = \sqrt{t}\cdot(1-\frac{1}{2}\ln t)$. This specific choice is used in Section 6 to determine an optimal economic capital formula for lookback financial losses. Section 7 applies this formula to the comparison of two risky investment strategies. In equity capital markets, which tend to move sideways, some investors express the opinion that the best investment strategy consists to “buy at the low and sell at the high”. Compared to a simple “buy and hold” strategy, this apparently better strategy requires much more economic capital. Even more, if one associates some specific cost of capital to investment strategies, then the “buy at the low and sell at the high” strategy might generate costs of capital, which cannot be financed by the expected return of the strategy and therefore render this strategy impossible to follow in practice.

The following notations and definitions are used throughout. Capital letters $X, Y,$ denote random variables with distribution functions $F_X(x), F_Y(x),$ and finite means μ_X, μ_Y. The survival function of X is denoted by $\overline{F}_X(x) = 1 - F_X(x)$. The stop-loss transform of a random variable X is defined by

$$\pi_X(x) := E[(X - x)_+] = \int_x^\infty \overline{F}_X(t)dt, \; x \text{ in the support of } X. \qquad (1.1)$$

The random variable X is said to precede Y in the usual *stochastic order*, a relation written as $X \leq_{st} Y$, if $\overline{F}_X(x) \leq \overline{F}_Y(x)$ for all x in the common support of X and Y. The random variables X and Y satisfy the *stop-loss order*, or equivalently, the *increasing convex order*, written as $X \leq_{sl} Y$ (or $X \leq_{icx} Y$), if $\pi_X(x) \leq \pi_Y(x)$ for all x. The partial order relations $\leq_{st}$ and $\leq_{sl}$ are widely used in probability theory and their applications (e.g., Kaas et al. [17], or Shaked and Shanthikumar [22]). The symbol $=_d$ denotes equality in distribution.

2. Coherent and Tail-Free Risk Measures

Let (Ω, A, P) be a probability space such that Ω is the space of outcomes or states of the world, A is the σ-algebra of events, and P is the probability measure. For a measurable real-valued random variable X on this probability space, that is a map $X : \Omega \to R$, the probability distribution of X is defined and denoted by $F_X(x) = P(X \leq x)$.

In the present paper, the random variable X represents a financial loss such that for $\omega \in \Omega$ the real number $X(\omega)$ is the realization of a loss and profit function with $X(\omega) \geq 0$ for a loss and $X(\omega) < 0$ for a profit. A set of financial losses is denoted by $\chi \cdot$ A *risk measure* is a functional from the set of losses to the extended non-negative real numbers described by a map $R : \chi \to [0, \infty]$. A *coherent risk measure* is a risk measure, which satisfies the following desirable properties (e.g., Artzner et al. [1, 2]):

(M) (monotonicity)**:** If $X, Y \in \chi$ satisfy $X \leq_{st} Y$, then $R[X] \leq R[Y]$.

(P) (positive homogeneity)**:** If $a > 0$ is a positive constant and $X \in \chi$, then $R[aX] = aR[X]$.

(S) (sub-additivity)**:** If $X, Y, X + Y \in \chi$, then $R[X + Y] \leq R[X] + R[Y]$.

(T) (translation invariance)**:** If c is a constant and $X \in \chi$, then $R[X + c] = R[X] + c$.

Definition 2.1. A continuous increasing function $g : [0, 1] \to [0, 1]$ such that $g(0) = 0$ and $g(1) = 1$ is called *distortion* function. For $X \in \chi$ with survival function $\overline{F}_X(x)$, the transform $\overline{F}_X^g(x) := g(\overline{F}_X(x))$ defines the so-called *distorted survival function*.

Taking the mean value with respect to the distorted survival function of a loss $X \in \chi$, one obtains the *distortion (risk) measure*

$$R_g[X] = \int_0^\infty \overline{F}_X^g(x)dx - \int_{-\infty}^0 \left[1 - \overline{F}_X^g(x)\right]dx. \tag{2.1}$$

Wang et al. [27], Theorem 3, implies that the risk measures (2.1) is a coherent risk measure provided $g(x)$ is a concave function.

Besides monotonicity, that is preservation of stochastic order, it is known that a distortion measure $R_g[X]$ with concave distortion function preserves the stop-loss order or increasing convex order of non-negative risks (e.g., Hürlimann [11]). This is a desirable property because increased risk should be penalized with an increased measure. With equal means and variances, a stop-loss order relation between different random variables cannot exist. In this situation, increased risk can be modelled by the degree three stop-loss order or equivalently, by equal mean and variance, the degree three convex order. Thus, one is interested in distortion measures, which preserve this higher degree convex order. As suggested by Yoshiba and Yamai [29], such measures should be called free of tail risk or simply tail-free distortion measures. Some more formal definitions and properties are required.

For any real random variable *X*, consider the higher order partial moments $\pi_X^n(x) = E\left[(X - x)_+^n\right]$, $n = 0, 1, 2, \ldots,$ called *degree n stop-loss transforms*. For $n = 0$, the convention is made that $(x - d)_+^0$ coincides with the indicator function $1_{\{x > d\}}$, hence $\pi_X^0(x) = \overline{F}_X(x)$ is simply the survival function of *X*. For $n = 1$, this is the usual stop-loss transform $\pi_X(x)$, written without upper index. It is not difficult to establish the recursion (e.g., Hürlimann [13], Theorem 2.1)

$$\pi_X^n(x) = n \cdot \int_x^\infty \pi_X^{n-1}(t)dt, \quad n = 1, 2, \ldots. \tag{2.2}$$

We consider the following variants of the higher degree stop-loss orders (see Kaas et al. [17], Hürlimann [13] among others).

Definition 2.2. For $n = 0, 1, 2, \ldots,$ a random variable *X* precedes *Y* in *degree n stop-loss transform order*, written $X \leq_{slt}^{(n)} Y$, if for all *x* one has $\pi_X^n(x) \leq \pi_Y^n(x)$. A random variable *X* precedes *Y* in *degree n stop-loss* *order*, written $X \le_{sl}^{(n)} Y$, if $X \le_{slt}^{(n)} Y$ and the moment inequalities $E[X^k] \le E[Y^k]$, $k = 1, \ldots, n-1$, are satisfied. With equal moments $E[X^k] = E[Y^k]$, $k = 0, \ldots, j$, for some $j \in \{0, \ldots, n\}$, the relation is written $X \le_{sl,j}^{(n)} Y$. In particular, the one extreme case $\le_{sl,0}^{(n)} \equiv \le_{sl}^{(n)}$ defines a general degree n stop-loss order and the other one $\le_{sl,n}^{(n)} \equiv \le_{(n+1)-cx}$ defines the so-called $(n+1)$-*convex order* recently studied by Denuit et al. [6]. Note that the special case $\le_{sl}^{(0)}$ is identical with the usual stochastic order $\le_{st}$. For $n = 1$, the stochastic order $\le_{sl}^{(1)}$ coincides with the usual stop-loss order $\le_{sl}$.

For fixed n, the above stop-loss order variants satisfy the following hierarchical relationship:

$$\le_{(n+1)-cx} \equiv \le_{sl,n}^{(n)} \Rightarrow \le_{sl,n-1}^{(n)} \Rightarrow \cdots \Rightarrow \le_{sl,1}^{(n)} \Rightarrow \le_{sl,0}^{(n)} \equiv \le_{sl}^{(n)} \Rightarrow \le_{slt}^{(n)}. \tag{2.3}$$

Moreover, the higher degree stop-loss orders build a *hierarchical class* of partial orders (Kaas et al. [17], Theorem 2.2), that is one has

$$\le_{sl,j}^{(n)} \Rightarrow \le_{sl,j}^{(n+1)}, \quad j \in \{0, \ldots, n\}. \tag{2.4}$$

Definition 2.3. A risk measure $R : \chi \to [0, \infty]$ is called a *degree n tail-free* if it is preserved under the $(n+1)$-convex order, that is, if $X, Y \in \chi$ satisfy $X \le_{(n+1)-cx} Y$, then $R[X] \le R[Y]$.

As mentioned above, it is known that a distortion measure $R_g[X]$ with concave distortion function preserves $\le_{(n+1)-cx}$ for $n = 0, 1$, and is thus a tail-free risk measure of degree zero and one. To motivate whether one is interested in specific concave distortion functions $g(x)$ such that

$R_g[X]$ is a degree two tail-free risk measure, it suffices to look at the following example, which is relevant in risk management (see also Hürlimann [15], Example 3.1).

Example 2.1. Conditional value-at-risk versus Wang right-tail measure.

Consider the coherent distortion measure (2.1) defined by the increasing concave distortion function $g_\varepsilon(x) = \min\left\{\frac{x}{\varepsilon}, 1\right\}$, where ε is a small probability of loss, say $\varepsilon = 0.05$. By definition, the measure associated to $X \in \chi$ is denoted by $R_{g_\varepsilon}[X]$. It is known that this risk measure coincides with several other known risk measures like the *conditional value-at-risk* measure and the *expected shortfall* measure (e.g., Hürlimann [14], Proposition 2.1). In standard notation, conditional value-at-risk at the confidence level $\alpha = 1 - \varepsilon$, written $CVaR_\alpha[X]$, coincides with $R_{g_\varepsilon}[X]$. For comparison, consider the distortion function $g(x) = \sqrt{x}$. The coherent distortion measure (2.1), called *Wang right-tail measure* and denoted by $WRT[X] := R_g[X]$, has been proposed by Wang [25] as a measure of right-tail risk. For illustration, let now Y be a loss consisting of two scenarios with loss amounts 20$, 2100$ such that $P(Y = 20) = 1 - P(Y = 2100) = \frac{25}{26}$. Through active risk management, assume that the lower amount can be eliminated and that the higher loss amount can be reduced to 1700$. With equal mean and variance, this results in a loss X such that $P(X = 0) = 1 - P(X = 1700) = \frac{16}{17}$. Suppose a risk manager is weighing the cost of risk management against the benefit of capital relief. Then CVaR does not promote risk management because $CVaR_\alpha[X] = 1700 > CVaR_\alpha[Y] = 20 + 2080 \cdot \left(\frac{20}{26}\right) = 1620$, which shows that there is a capital penalty instead of a capital relief for either removing or reducing the initial loss amounts. However, the Wang right-tail measure offers a capital relief because $WRT[X] = 1700 \cdot \sqrt{\frac{1}{17}} = 412.3 < WRT[Y] = 20 + 2080 \cdot \sqrt{\frac{1}{26}} = 427.9$. Since Y is evidently a higher loss than X, the CVaR measure fails to recognize this feature. Even more, in this example, X precedes Y in the degree three convex order. This shows through a meaningful counterexample that CVaR is not a degree two tail-free coherent risk measure.

3. The Blackwell-Dubins-Gilat-Kertz-Rösler Theorem on Maxima of Martingales

In the present section, the continuum of real random variables $\{X(t)\}_{0 \le t \le 1}$, defined with respect to the probability space (Ω, F, P), represents the evolution of a financial loss over a time period of length one. More precisely, the random variable $X(t)$ represents a financial loss such that for $\omega \in \Omega$ the real number $X(t, \omega)$ is the realization of a loss and profit function with $X(t, \omega) \ge 0$ for a loss and $X(t, \omega) < 0$ for a profit. We suppose that $\{X(t)\}_{0 \le t \le 1}$ is a *martingale*. Therefore, there exists a filtration $\{F_t\}_{0 \le t \le 1}$ such that

(i) $\{X(t)\}$ is $\{F_t\}$-adapted,

(ii) $X(t)$ is integrable for any $0 \le t \le 1$ and $E[X(t) | F_s] = X_s$ for any $0 \le s < t \le 1$,

(iii) the paths $t \to X(t)$ are right-continuous and have left-hand limits.

The maximal random variable $M = \sup_{0 \le t \le 1} \{X(t)\}$ represents the maximum financial loss over one time period, called here *lookback financial loss*. By convention, the terminal random variable, which represents the financial loss at expiry, is denoted by $X = X(1)$ and has the distribution $F_X(x)$. It is assumed throughout that $E[|X|]$ exists and is finite. Recall that the *Hardy-Littlewood transform* X^H of X is defined by its quantile function on $[0, 1]$ through the formula

$$(F_X^H)^{-1}(u) = \begin{cases} \frac{1}{1-u}\int_u^1 F_X^{-1}(\nu)d\nu, & u < 1, \\ F_X^{-1}(1), & u = 1. \end{cases} \tag{3.1}$$

Its name stems from the Hardy-Littlewood [9] maximal function. The random variable X^H is the least majorant with respect to $\leq_{st}$ among all random variables $Y \leq_{sl} X$ (e.g., Meilijson and Nàdas [20]). Its great importance in applied probability and related fields has been noticed by several further authors, among others Blackwell and Dubins [3], Dubins and Gilat [7], Rüschendorf [21], and Kertz and Rösler [18, 19].

The following result identifies the set of maxima of martingales $M = \sup_{0 \leq t \leq 1}\{X(t)\}$ with the set of random variables stochastically bounded below by X and above by X^H.

Theorem 3.1 (BDGKR-theorem on maxima of martingales)**.** *Let X be a random variable such that $E[|X|] < \infty$ and let X^H be its Hardy-Littlewood transform. Then the following sets of random variables coincide*:

$\{M$: *there is a martingale* $\{X(t)\}_{0 \leq t \leq 1}$ *such that* $X(1) =_d X$ *and* $M =_d \sup_{0 \leq t \leq 1}\{X(t)\}\}$

$$= \left\{M : X \leq_{st} M \leq_{st} X^H\right\}.$$

Proof. The inclusion "$\subset$" has been settled by Dubins and Gilat [7], Theorem 1, where the partial result $M \leq_{st} X^H$ is a reformulation of Blackwell and Dubins [3], Theorem 3, assertion (a). The reversed inclusion "$\supset$" has been derived later by Kertz and Rösler [18]. For details, the reader should consult the mentioned papers. □

Remark 3.1. As a refinement, Kertz and Rösler [18], Theorem 3.4, also characterize the set of random variables M such that for given random variables X and Y, there exists a martingale $\{X(t)\}_{0 \le t \le 1}$ such that $X(0) =_d X$, $X(1) =_d Y$, and $M =_d \sup_{0 \le t \le 1} \{X(t)\}$. From a similar but more recent result by Hobson and Pedersen [10], one knows that there exists a martingale $\{X(t)\}_{0 \le t \le 1}$ such that $X(0) =_d X$, $X(1) =_d Y$, and $M =_d \inf_{0 \le t \le 1} \{X(t)\}$. To remain brief, these more general situations are, however, not considered in the present paper.

4. An Axiomatic Approach to the Lookback Distortion Measure

We rely on the axiomatic characterization of the coherent distortion measures by Wang et al. [27], which is briefly reviewed. Adding some extra condition, we show that the lookback distortion measure, defined by $g(t) = t^{\rho} \cdot [1 - \rho \cdot \ln(t)]$, $\rho \in (0, 1]$, is a meaningful coherent distortion measure for use in the context of lookback financial losses. The following axioms have been proposed:

(A1) (conditional state independence)**:** The risk measure $R[X]$ of a loss $X \in \chi$ depends only on its distribution function $F_X(x) = P(X \le x)$.

(A2) (monotonicity)**:** If $X, Y \in \chi$ satisfy $X \le_{st} Y$, then $R[X] \le R[Y]$.

(A3) (comonotone additivity)**:** If $X, Y \in \chi$ are comonotone, that is, there exists $Z \in \chi$ and increasing real functions f and g such that $X = f(Z)$, $Y = g(Z)$, then $R[X + Y] = R[X] + R[Y]$.

(A4) (continuity)**:** For $X \in \chi$, one has $\lim_{d \to -\infty} R[\max(X, d)] = R[X]$, $\lim_{d \to \infty} R[\min(X, d)] = R[X]$. Furthermore, if $X \ge 0$ and $d \ge 0$, one has $\lim_{d \to 0^+} R[(X - d)_+] = R[X]$.

Special types of losses, which will play a crucial role in the next and following results, are the Bernoulli random losses B_p, $p \in [0, 1]$, with $p = P(B_p = 1) = 1 - P(B_p = 0)$, as well as *positive affine transforms* thereof, that is random variables of the type $cB_p + a$, with $c > 0$, which define the biatomic losses. For coherent risk measures, the sub-additive property (S) of Section 2 is also assumed. One has the following remarkable result.

Theorem 4.1 (Characterization of coherent distortion measures)**.** *Assume* χ *contains all Bernoulli losses* B_p, $p \in [0, 1]$. *The risk measure* $R : \chi \to [0, \infty]$ *satisfies the axioms* (A1)-(A4), *the property* $R[1] = 1$ *and sub-additivity if*, *and only if, there exists a continuous increasing concave distortion function* $g(x)$ *such that* $R[X] = R_g[X]$ *for all* $X \in \chi$.

Proof. This follows from Theorem 3 and the Appendix A in Wang et al. [27]. □

We are now interested in coherent distortion measures, which could be used in the context of lookback financial losses. According to Theorem 3.1, the terminal random value $X = X(1)$ of a martingale $\{X(t)\}_{0 \le t \le 1}$ and its Hardy-Littlewood transform X^H characterize the set of all possible lookback financial losses $M = \sup_{0 \le t \le 1} \{X(t)\}$. Suppose two partners A and B conclude the following agreement. Partner A pays B the risk measure equivalent $R[M]$ and receives in exchange the random payment M from B. What is an appropriate risk measure for lookback financial losses?

Firstly, since $X \le_{st} M \le_{st} X^H$ by Theorem 3.1, it is reasonable to agree that the risk measure for the actual lookback financial loss should not depend on M itself, but rather on the stochastic upper bound X^H. Secondly, with the defining formula (3.1), the transform X^H depends only on X. Therefore, an appropriate risk measure for M should only depend on the random terminal value X of the martingale associated to M, that is $R[M] := R[X]$. In the other words, the set χ of losses used in the above axiomatic approach consists of the terminal values $X = X(1)$ of the martingales $\{X(t)\}_{0 \le t \le 1}$ associated to the lookback financial losses $M = \sup_{0 \le t \le 1} \{X(t)\}$. Since the upper bound X^H is attained for some maximum of martingale, we require as extra condition the following inequality:

$$\text{(A5)} \qquad R[X] \ge E\lfloor X^H \rfloor = R_g[X],$$

where $g(x) = x \cdot (1 - \ln x)$ (use Kertz and Rösler [18], formula (4.1)). Thirdly, for prudent valuation, the inequality in (A5) should be strict. A one-parameter family of distortion functions, which fulfills this requirement, is the lookback distortion function, which satisfies by Theorem 4.1 the following axiomatic characterization:

Corollary 4.1 (Coherent distortion measure for lookback financial losses)**.** *Let the set* χ *of losses consists of the terminal values* $X = X(1)$ *of the martingales* $\{X(t)\}_{0 \le t \le 1}$ *associated to the lookback financial losses* $M = \sup_{0 \le t \le 1} \{X(t)\}$. *Assume* χ *contains all Bernoulli losses* B_p, $p \in [0, 1]$. *Then, the lookback distortion measure* $R[X] = R_g[X]$, $X \in \chi$, *with* $g(t) = t^{\rho} \cdot [1 - \rho \cdot \ln(t)]$, $\rho \in (0, 1]$, *satisfies the axioms* (A1)-(A5), *the property* $R[1] = 1$ *and sub-additivity.*

Remark 4.1. A lookback distortion function has been first introduced in Hürlimann [12] (see also Hürlimann [15], Example 4.2).

5. Tail-Free Properties of the Lookback Distortion Measure

To prevent counterexamples of the type presented in Example 2.1, we derive a condition on the distortion function, which guarantees that the measure defined in (2.1) is a degree two tail-free distortion measure when restricted to the subset of biatomic losses with equal mean and variance. For this, one must show that the distortion measure is preserved under the degree three convex order.

Let D_2 denote the subset of all real-valued standard biatomic random variables with mean zero and variance one. Recall that an element $X \in D_2$ is uniquely determined by its support $\{\bar{x}, x\}$ with $x > 0$, $\bar{x} = -x^{-1}$, and probabilities $P(X = x) = 1 - P(X = \bar{x}) = (1 + x^2)^{-1}$. It is convenient to identify X with its support and use the short-hand notation $X = \{\bar{x}, x\}$ (e.g., Hürlimann [16], Theorem I.5.1).

Lemma 5.1. *Let* $X = \{\bar{x}, x\}$ *and* $Y = \{\bar{y}, y\}$ *belong to* D_2. *Then one has* $X \le_{3-cx} Y$ *if, and only if, one has* $\bar{x} < \bar{y} < 0 < x < y$.

Proof. If $X \le_{3-cx} Y$, then one has necessarily $E\lfloor X^3 \rfloor = \bar{x} + x < E\lfloor Y^3 \rfloor = \bar{y} + y$, hence $\bar{x} < \bar{y} < 0 < x < y$. Conversely, if the latter inequalities hold, one has $E\lfloor X^3 \rfloor < E\lfloor Y^3 \rfloor$ and the difference $F_X(x) - F_Y(x)$ in distributions has two sign changes in the order $(+, -, +)$. The affirmation $X \le_{3-cx} Y$ follows from Denuit et al. [6], Theorem 4.3. □

An arbitrary biatomic loss with mean μ and standard deviation σ is a positive affine transform $\mu + \sigma \cdot X$ with $X = \{\bar{x}, x\} \in D_2$. The distortion measure (2.1) satisfies the relationship $R_g[\mu + \sigma \cdot X] = \mu + \sigma \cdot R_g[X]$ with

$$R_g[X] = \frac{1 + x^2}{x} \cdot g\left(\frac{1}{1 + x^2}\right) - \frac{1}{x}. \tag{5.1}$$

Applying Lemma 5.1, it follows that the distortion measure $R_g[X]$ is tail-free of degree two for the subset of biatomic losses if, and only if, the function of one variable in (5.1) is monotone increasing.

Proposition 5.1. *Let* $g(x)$ *be a continuous and differentiable increasing concave distortion function. The coherent distortion measure* $R_g[X]$ *is a degree two tail-free risk measure for the subset of biatomic losses if, and only if, the following condition holds*:

$$\frac{t}{1-t}[1-g(t)]+g(t)-2tg'(t)\geq 0, \quad t\in(0,1). \tag{5.2}$$

Proof. The function $h(x)=\frac{1+x^2}{x}\cdot g\left(\frac{1}{1+x^2}\right)-\frac{1}{x}$ is monotone increasing if, and only if, one has $\frac{1}{x^2}\left[1-g\left(\frac{1}{1+x^2}\right)\right]+g\left(\frac{1}{1+x^2}\right)-\frac{2}{1+x^2}g'\left(\frac{1}{1+x^2}\right)\geq 0$, $x>0$. Making the change of variables $t=(1+x^2)^{-1}$ this condition identifies with (5.2). □

Among the many concave distortion functions known in the literature, only a few turn out to generate degree two tail-free coherent risk measures for the subset of biatomic losses. As a first example, the PH-distortion measure, defined by $g(t)=t^{\rho}$, $\rho\in(0,1]$, is degree two tail-free for the subset of biatomic losses if, and only if, one has $\rho\in(0,\frac{1}{2}]$ or $\rho=1$ (Hürlimann [15], Proposition 4.2). A similar result holds for the lookback distortion measure, and completes herewith the statement made in Hürlimann [15], Example 4.2.

Proposition 5.2. *Let* $g(t)=t^{\rho}\cdot[1-\rho\cdot\ln(t)]$, $\rho\in(0,1]$, *be the lookback distortion function. The lookback distortion measure is degree two tail-free for the subset of biatomic losses if, and only if, one has* $\rho\in(0,\frac{1}{2}]$.

Proof. First, we show that for $\rho \in (0, \frac{1}{2}]$ the lookback distortion measure is a degree two tail-free coherent risk measure for the subset of biatomic losses. Denote by $H(t)$ the left-hand side in condition (5.2). Using that $\rho \leq \frac{1}{2}$, a calculation shows the inequality

$$H(t) \geq t \cdot \left\{1 + t^{\rho-1} \cdot [1 - 2t + \frac{1}{2} t \cdot \ln(t)]\right\}.$$

It suffices to show that the curly bracket is non-negative. Setting $t = 1 - u$, this is equivalent to the inequality

$$G(u) = (1-u)^{\rho-1} \cdot \{1 - 2u + \frac{1}{2}(1-u) \cdot \ln(\frac{1}{1-u})\} \leq 1, \quad u \in (0, 1).$$

Using the series expansion $-\ln(1-u) = \sum_{k=1}^{\infty} \frac{1}{k} u^k$, one obtains that

$$G(u) = (1-u)^{\rho-1} \cdot \left\{1 - u - \frac{1}{2}\left(u + \sum_{k=1}^{\infty} (\frac{1}{k} - \frac{1}{k+1}) u^{k+1}\right)\right\} \leq (1-u)^{\rho} \leq 1,$$

which shows the desired inequality. Conversely, we show that the lookback distortion measure is not degree two tail-free in case $\rho \in (\frac{1}{2}, 1)$. Let $Y = \left\{2\overline{x}, \frac{1}{2}x\right\}$, $X = \{\overline{x}, x\}$ be biatomic losses in D_2 such that $Y \leq_{3-cx} X$. Using (5.1), we show that there exists $x > 0$ such that

$$R_g[Y] - R_g[X] = \frac{4+x^2}{2x} \cdot g\left(\frac{4}{4+x^2}\right) - \frac{1+x^2}{x} \cdot g\left(\frac{1}{1+x^2}\right) - \frac{1}{x} > 0. \quad (5.3)$$

With the substitution $t = (1 + x^2)^{-1}$, it suffices to show that there exists $t > 0$ such that

$$h(t) = \frac{1}{\sqrt{1-t}}$$

$$\cdot \left\{2^{2\rho-1}(1+3t)^{1-\rho} t^{\rho - \frac{1}{2}}\left[1 - \rho \ln\left(\frac{4t}{1+3t}\right)\right] - t^{\rho-\frac{1}{2}}[1 - \rho \ln(t)] - \sqrt{t}\right\},$$

is strictly positive. Since $\ln\left(\frac{1+3t}{t}\right) \geq \ln\left(\frac{1}{t}\right)$ for $t \in (0, 1]$, one obtains

$$h(t) \geq \frac{t^{\rho-\frac{1}{2}}}{\sqrt{1-t}} \cdot \left\{\left(2^{2\rho-1}-1\right) \cdot (1-\rho \ln t) - \rho 2^{2\rho-1}(1+3t)^{1-\rho} \ln(4) - t^{1-\rho}\right\}.$$

Clearly, for $\rho \in (\frac{1}{2}, 1)$, one has $2^{2\rho-1} - 1 > 0$. Therefore, for t sufficiently small, say $t \in (0, t_0)$, the curly bracket is strictly positive. It follows that $h(t) > 0$ for all $t \in (0, t_0)$, which shows the desired assertion. □

Next, comparing Weibull and Pareto losses with equal means and variances, we show that the lookback distortion measure is degree two tail-free if, and only if, one has $\rho = \frac{1}{2}$.

As a preliminary step, we show that by equal finite mean and variance, the Weibull and Pareto distributions are ordered in the degree three convex order. Scaling the distributions, we assume without loss of generality that the equal mean is one, and that the finite equal coefficient of variation K satisfies the inequality $K > 1$. A *Weibull* distributed random variable X with mean one has survival function $\overline{F}_X(x) = e^{-\lambda x^c}$ and density

$$f_X(x) = \lambda c x^{c-1} \exp(-\lambda x^c), \quad x > 0, \quad 0 < c < 1, \quad \lambda = \Gamma\left(1 + c^{-1}\right)^c. \tag{5.4}$$

The coefficient of variation satisfies the relationship:

$$1 + K^2 = \Gamma\left(1 + 2c^{-1}\right) \cdot \Gamma\left(1 + c^{-1}\right)^{-2}. \tag{5.5}$$

A *Pareto* distributed random variable Y with mean one has survival function $\overline{F}_Y(x) = \left(1 + \frac{x}{a}\right)^{-\gamma}$ and density

$$f_Y(x) = \frac{\gamma}{a}\left(1 + \frac{x}{a}\right)^{-(\gamma+1)}, \; x > 0, \; a = \frac{K^2+1}{K^2-1} \cdot \mu, \; \gamma = 2 \cdot \frac{K^2}{K^2-1}. \tag{5.6}$$

Proposition 5.3. *By equal finite mean and variance, the Weibull precedes the Pareto in the degree three convex order, that is,* $X \le_{3-cx} Y$.

Proof. Consider the difference in logarithmic derivatives of the densities

$$\rho(x) = \frac{d}{dx}\ln f_Y(x) - \frac{d}{dx}\ln f_X(x) = \frac{h(x)}{x}, \text{ with } h(x) = \lambda c x^c + 1 - c - \frac{(\gamma+1)x}{\lambda + x}.$$

Since $h(0) = 1 - c > 0$, $h(\infty) = \infty$, and $h'(x)$ has two zeros (local extrema of $h(x)$), one sees that $h(x)$ and $\rho(x)$ have two sign changes in the order $(+, -, +)$. The affirmation follows from Denuit et al. [6], Theorem 4.6.

□

We derive now explicit formulas for the lookback distortion measures of the Weibull and Pareto, and show through comparison that the degree three convex order is only preserved in the special case $\rho = \frac{1}{2}$. For the Weibull, one has

$$R_g[X] = \int_0^\infty g(\overline{F}_X(x))dx = \int_0^1 Q_X(1-u)g'(u)du = \int_0^\infty e^{-\lambda\rho x^c}dx + \int_0^\infty \lambda\rho x^c e^{-\lambda\rho x^c}dx$$

$$= (\lambda\rho)^{-c^{-1}} c^{-1}\Gamma(c^{-1}) + (\lambda\rho)^{-c^{-1}} c^{-1}\Gamma(1 + c^{-1}) = \rho^{-c^{-1}}(1 + c^{-1}). \quad (5.7)$$

Similarly, the Pareto has quantile function $Q_Y(u) = a \cdot \left((1-u)^{-\gamma^{-1}} - 1\right)$, $u \in (0, 1)$. Using that $-c^2 x^{c-1}\ln(x)$ is the derivative of $x^c(1 - c\ln(x))$, one obtains

$$R_g[Y] = \int_0^1 Q_Y(1-u)g'(u)du = a\rho^2 \int_0^1 \left(u^{\rho-1} - u^{\frac{\rho\gamma-1}{\gamma}-1}\right)\ln(u)du$$

$$= a\left(\left(\frac{\rho\gamma - 1}{\gamma}\right)^2 - 1\right) = \left(\frac{K^2+1}{K^2-1}\right)\cdot\left(\left(\frac{2\rho K^2}{(2\rho-1)K^2+1}\right)^2 - 1\right). \quad (5.8)$$

One must have $R_g[Y] \geq E[Y]-1,$ which is feasible provided

$$\rho \geq \frac{1}{\gamma} \cdot \frac{\sqrt{1+\lambda^{-1}}}{\sqrt{1+\lambda^{-1}}+1} = \frac{1}{2}\left(\frac{K^2-1}{K^2}\right) \cdot \left(\frac{\sqrt{2}K}{\sqrt{2}K+\sqrt{K^2+1}}\right). \tag{5.9}$$

Proposition 5.4. *The lookback distortion measure* $g(t)=t^{\rho} \cdot [1-\rho \cdot \ln(t)],$ $\rho \in (0, 1],$ *is degree two tail-free for the subset of all Weibull and Pareto losses with equal finite means and variances if, and only if, one has* $\rho = \frac{1}{2}.$

Proof. First, for each $\rho \neq \frac{1}{2},$ we show that there exists $c \in (0, 1)$ and a corresponding parameter $K^2 = \Gamma\left(1+2c^{-1}\right) \cdot \Gamma\left(1+c^{-1}\right)^{-2} - 1$ such that $R_g[X] > R_g[Y].$ Since $X \leq_{3-cx} Y$ by Proposition 5.3, this is a counterexample to the tail-free property.

Case 1.

$$\rho \geq \frac{1}{2+\sqrt{2}} = \lim_{k \to \infty} \frac{1}{2}\left(\frac{K^2-1}{K^2}\right) \cdot \left(\frac{\sqrt{2}K}{\sqrt{2}K+\sqrt{K^2+1}}\right).$$

One has $\lim_{k \to \infty} R_g[Y] = \left(\frac{2\rho}{2\rho-1}\right)^2 - 1 \geq 1.$ Therefore, given an arbitrary small $\varepsilon > 0,$ there exists $c_0 \in (0, 1)$ such that for all $c \in (0, c_0)$ one has $R_g[Y] < \left(\frac{2\rho}{2\rho-1}\right)^2 - 1 + \varepsilon.$ Since it is always possible to choose $c_1 \leq c_0$ such that for all $c \in (0, c_1),$ one has

$R_g[X] - R_g[Y] = \rho^{-c^{-1}}(1+c^{-1}) - \left(\frac{2\rho}{2\rho-1}\right)^2 + 1 - \varepsilon > 0,$ the tail-free property does not hold.

Case 2.

$$\rho < \frac{1}{2+\sqrt{2}}.$$

For the choice of K, which solves the equation $\rho = \frac{1}{2}\left(\frac{K^2-1}{K^2}\right)$. $\left(\frac{\sqrt{2}K}{\sqrt{2}K + \sqrt{K^2+1}}\right)$, one has $R_g[Y] = 1$. For the corresponding parameter $c \in (0, 1)$ such that $\Gamma\left(1+2c^{-1}\right)\cdot\Gamma\left(1+c^{-1}\right)^{-2} = 1 + K^2$, one has clearly $R_g[X] = \rho^{-c^{-1}}(1+c^{-1}) > 1 = R_g[Y]$, which again disproves the tail-free property.

It remains to show that for $\rho = \frac{1}{2}$ the tail-free property holds. For this, one must show that the function $f(c) = \Gamma\left(1+2c^{-1}\right)^2 \cdot \Gamma\left(1+c^{-1}\right)^{-4} - 2^{c^{-1}}(1+c^{-1})$ is non-negative for all $c \in (0, 1)$. Since $f(c)$ is monotone decreasing on $[0, 1]$, one obtains $f(c) > f(1) = 0$ for all $c \in (0, 1)$, which shows the affirmation. □

As motivated in Section 2, a sound risk measure, which provides incentive for risk management, should satisfy the following axiom:

(A6) (degree two tail-free condition)**:** If $X, Y \in \chi$ satisfy a degree three convex order relation $X \leq_{3-cx} Y$, then $R[X] \leq R[Y]$.

As a main result of our axiomatic approach, only a very specific choice of the lookback distortion function is suitable for risk measurement in the context of lookback financial losses.

Theorem 5.1 (Characterization of specific lookback distortion measure)**.** *Let the set* χ *of losses consists of the terminal values* $X = X(1)$ *of the martingales* $\{X(t)\}_{0\leq t\leq 1}$ *associated to the lookback financial losses*

$M = \sup_{0 \le t \le 1} \{X(t)\}$. *Assume* χ *contains all Bernoulli losses and positive affine transforms thereof, as well as all Weibull and Pareto losses with finite means and variances. Then the lookback distortion measure* $R_g[X]$, $X \in \chi$, *with* $g(t) = t^{\rho} \cdot [1 - \rho \cdot \ln(t)]$, $\rho \in (0, 1]$, *satisfies the axioms* (A1)-(A6), *the property* $R[1] = 1$ *and sub-additivity if, and only if, one has* $\rho = \frac{1}{2}$.

Proof. This follows immediately using Theorem 4.1, Propositions 5.2 and 5.3. □

Remark 5.1. It is not known for which largest set χ the specific lookback distortion measure derived in Theorem 5.1 is still tail-free of degree two, and thus satisfies (A6). However, numerical evidence shows that, for many of the common two-parameter continuous distributions, the axiom (A6) is fulfilled. It would be worthwhile to construct some simple counterexample, if possible.

6. Economic Capital for Lookback Financial Losses

The use of distortion measures to determine capital requirements of a risky business can be found in many papers (e.g., Wirch and Hardy [28], Wang [26], Goovaerts et al. [8]). We follow Goovaerts et al. [8], Example 10 (see also Hürlimann [15], Section 7).

Let $M = \sup_{0 \le t \le 1} \{X(t)\}$ represent the lookback financial loss (e.g., $X(t)$ is the invested amount minus its accumulated return at time $t \in [0, 1]$) at the end of a one-year period. To avoid the insolvency risk, an investor borrows at the beginning of the period and at the interest rate i some economic capital $C = EC[X]$, which is assumed to depend only on the terminal value $X = X(1) \in \chi$ of the martingale $\{X(t)\}_{0 \le t \le 1}$ associated to M. This capital is invested at the risk-free interest rate $r < i$. The resulting (net) *interest on capital* $(i-r)C$ should be as small as possible. On the other hand, insolvency occurs if $M > C(1+r)$, hence C should be as large as possible. Therefore, an "optimal" compromise solution must be found. Theoretically, to eliminate the solvability risk, the investor could buy on the financial market (if available) an option with payoff $(M-C(1+r))_+$. If the price of such a contract is set using a risk measure, then the *cost of solvability* equals $R[(M-C(1+r))_+]$. The aggregate cost of solvability and interest on capital determines the *cost of capital function* $f(C) = R[(M-C(1+r))_+] + (i-r)C$, which should be minimized. Assume market prices are set using a coherent distortion measure such that $R[M] = R_g[X]$ for all $X \in \chi$, as justified in Section 4. Using the distorted survival function $\overline{F}_X^g(x) = g(\overline{F}_X(x))$ associated to the survival function of X, the cost of capital function can be rewritten as

$$f(C) = E^g[(X-C(1+r))_+] + (i-r)C, \tag{6.1}$$

where $E^g[X]$ denotes expectation of X under the distorted survival function. Under the assumption of *continuous* distributions, the *optimal economic capital*, which minimizes (6.1), and the corresponding *minimum cost of capital* are determined as follows (e.g., (6.2) is formula (6) in Goovaerts et al. [8]):

$$EC[X] = \frac{1}{1+r}\left(\overline{F}_X^g\right)^{-1}\left(\frac{i-r}{1+r}\right) = \frac{1}{1+r}F_X^{-1}\left(1-g^{-1}\left(\frac{i-r}{1+r}\right)\right), \tag{6.2}$$

$$f(EC[X]) = \frac{i-r}{1+r}E^g[X|X > (1+r)EC[X]]. \tag{6.3}$$

The formula (6.2) identifies the end of the period value of the optimal economic capital, that is, $(1+r)EC[X]$, as the *value-at-risk* of X at the confidence level $\alpha = 1 - g^{-1}\left(\frac{i-r}{1+r}\right)$, that is, $(1+r)EC[X] = VaR_\alpha[X]$ in the usual notation. Similarly, the end of the period value of the minimum cost of capital identifies with the interest at the net rate $(i - r)$ on the *distorted conditional value-at-risk* of X at the same confidence level evaluated with respect to the distorted survival function, that is $(i - r)E^{g}[X|X > VaR_{\alpha}[X]] = (i - r)CVaR_{\alpha}^{g}[X]$, where the latter notation remembers the usual notation of conditional value-at-risk. In this setting, the considerable amount of recent research on related matters remains applicable.

Furthermore, if market prices are set using the axioms (A1)-(A6), if χ contains Bernoulli losses and positive affine transforms thereof, as well as Weibull and Pareto losses with finite means and variances, and if the lookback distortion function is applied, then Theorem 5.1 implies the unique solution $g(t) = \sqrt{t} \cdot \left(1 - \frac{1}{2}\ln t\right)$. With this choice, the *optimal confidence level* α for the capital requirement of lookback financial losses solves the equation

$$\sqrt{1 - \alpha} \cdot \left(1 - \frac{1}{2}\ln(1 - \alpha)\right) = \frac{i - r}{1 + r}. \tag{6.4}$$

7. Comparison of Two Investment Strategies

To illustrate our results, we compare capital requirements for the "buy and hold" and the "buy low and sell high" investment strategies over a time period of length one. Suppose that the financial loss is represented by the martingale $\{L(t)\}_{0 \le t \le 1}$, where $L(t) = e^{\delta t} - X(t)$, with $\delta = \ln(1 + r)$ the instantaneous risk-free return and $X(t)$ is the accumulated random return. For the evaluation of capital requirements, we use the end of period random loss $L = L(1) = e^{\delta} - X(1) = 1 + r - X$ for the "buy and hold" strategy and the maximum random loss $M = \sup_{0 \le t \le 1}\{L(t)\}$ for the "buy low and sell high" strategy, where only the terminal value is relevant in calculations, as seen in Section 6. We assume that X is lognormally distributed with parameters $\mu = \ln(1+r) - \frac{1}{2}\sigma^2$ and σ, where σ is the volatility. Therefore, the distribution of X is $F_X(x) = \Phi\left(\frac{\ln x - \mu}{\sigma}\right)$, with $\Phi(x)$ the standard normal distribution.

As motivated and justified in Hürlimann [15], Section 7, capital requirements for the "buy and hold" strategy are evaluated with the formulas (6.2) and (6.3) using $g(t) = \sqrt{t}$, which generates the right-tail measure of Wang [25]. For the "buy low and sell high" strategy, we use $g(t) = \sqrt{t} \cdot (1 - \frac{1}{2}\ln t)$ as justified in Section 6. One obtains the following formulas for economic capital (EC) and cost of capital (CoC):

$$EC = 1 - \frac{1}{1+r}\exp\left(\mu + \sigma\Phi^{-1}(\varepsilon)\right), \tag{7.1}$$

$$CoC = \frac{i-r}{1+r} \cdot \left\{(1+r)EC + \frac{1}{\varepsilon}\int_{(1+r)EC}^{1+r} g\left[\Phi\left(\frac{\ln(1+r-x)-\mu}{\sigma}\right)\right]dx\right\}, \tag{7.2}$$

where $\varepsilon = 1 - \alpha = \left(\frac{i-r}{1+r}\right)^2$ for the "buy and hold" strategy, and $\sqrt{\varepsilon}(1 - \frac{1}{2}\ln\varepsilon) = \frac{i-r}{1+r}$ for the "buy low and sell high" strategy.

Numerical results are displayed in Table 7.1. The increased riskiness of the "buy low and sell high" strategy is evident. Economic capital is substantially higher and, for the proposed valuation method, cost of capital explodes and cannot be financed by the expected return of the strategy, which shows that this strategy is not feasible in practice.

Table 7.1. Percentages of economic capital and cost of capital for two investment strategies

Percentages			"buy and hold" strategy		"buy low and sell high" strategy	
r	i	σ	EC	CoC	EC	CoC
4	6	10	29.00	4.06	35.47	145.65
	8	15	36.69	6.42	45.19	147.24
	10	20	43.03	8.72	51.58	109.69
5	6	10	31.57	3.46	37.33	144.60
	8	15	38.35	5.84	46.48	153.41
	10	20	44.44	8.11	54.29	155.28
6	8	15	40.49	5.16	48.43	175.30
	10	20	46.03	7.44	55.46	160.03

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